

Dot product

For $\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\vec{b} = \langle b_1, b_2, b_3 \rangle$ two vectors,

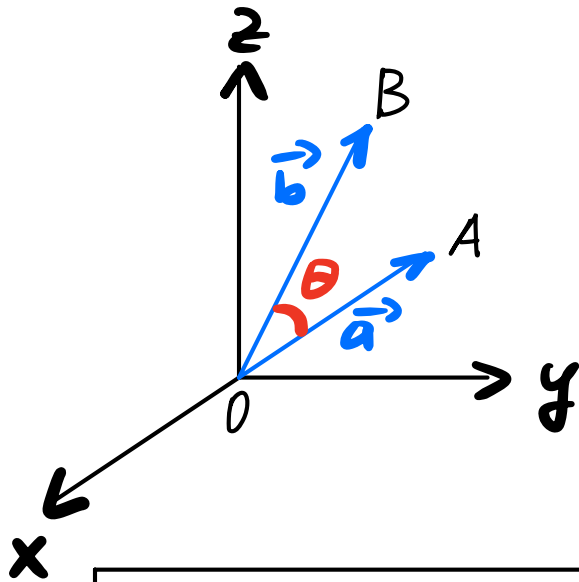
$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

the dot product
(or scalar product, or inner pr.)

Rmk: $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

Thm: $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$

$\uparrow$ lengths (magnitudes) $\uparrow$ angle



Corollary:

For $\vec{a}, \vec{b}$ non-zero vectors

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$$

// non-zero $\vec{a}, \vec{b}$ orthogonal
 $\Leftrightarrow \vec{a} \cdot \vec{b} = 0$ //

A diagram showing a vector $\vec{a}$ (red) and a vector $\vec{b}$ (blue). A dashed line from the tip of $\vec{a}$ perpendicular to $\vec{b}$ indicates the projection. The angle between them is θ .

$$|\text{comp}_{\vec{b}} \vec{a}| = \frac{|\vec{a} \cdot \vec{b}|}{|\vec{b}|}$$

Cross product

For $\vec{a} = \langle a_1, a_2, a_3 \rangle$ two vectors, the vector
 $\vec{b} = \langle b_1, b_2, b_3 \rangle$

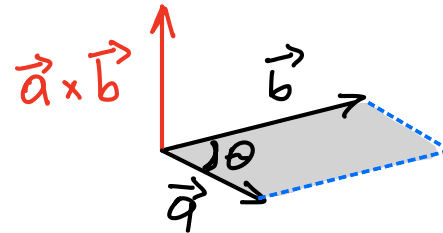
Works
only in $\mathbb{R}^3$

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

is the cross product (or vector product)

Thm: $\vec{a} \times \vec{b}$ is orthogonal
to both $\vec{a}$ and $\vec{b}$

(direction of $\vec{a} \times \vec{b}$ is given
by right hand rule)



Thm: $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$

↑
area of the
parallelogram

↑ angle between
 $\vec{a}$ and $\vec{b}$

- $\vec{a}$ and $\vec{b}$ are parallel iff $\vec{a} \times \vec{b} = 0$ ($\vec{a} \times \vec{a} = 0$)

Properties of cross product:

$$1) \vec{i} \times \vec{j} = \vec{k}, \quad \vec{j} \times \vec{k} = \vec{i}, \quad \vec{k} \times \vec{i} = \vec{j} \\ \vec{j} \times \vec{i} = -\vec{k}, \quad \vec{k} \times \vec{j} = -\vec{i}, \quad \vec{i} \times \vec{k} = -\vec{j}$$

$$2) \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

$$3) \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

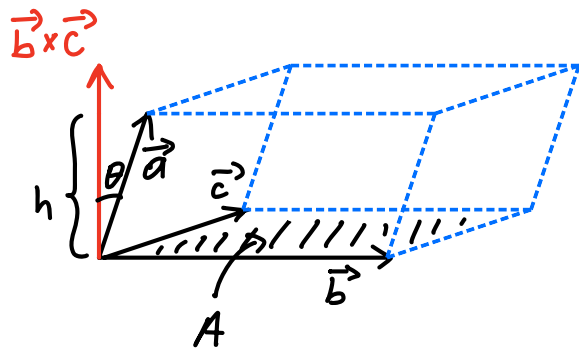
Rmk:

$$(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$$

$$(\vec{i} \times \vec{i}) \times \vec{j} = \vec{0} \quad \vec{i} \times (\underbrace{\vec{i} \times \vec{j}}_{\vec{k}}) = -\vec{j}$$

Triple product:

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} - \text{(scalar) triple product of } \vec{a}, \vec{b}, \vec{c}$$



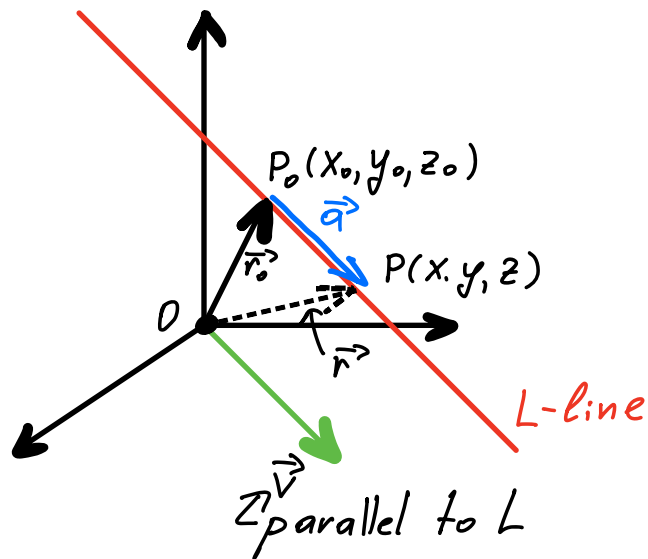
Volume of the parallelepiped

$$V = A \cdot h = |\vec{b} \times \vec{c}| (|\vec{a}| \cos \theta) \\ = |\vec{a} \cdot (\vec{b} \times \vec{c})|$$

Rmk: If $V = 0$ then $\vec{a}, \vec{b}, \vec{c}$

lie in the same plane (are co-planar)

Lines



$$\vec{r} = \vec{r}_0 + t\vec{v}$$

$t \in \mathbb{R}$
parameter

- vector equation of L

• Let $\vec{v} = \langle a, b, c \rangle$

then in components:

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

$$\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}$$

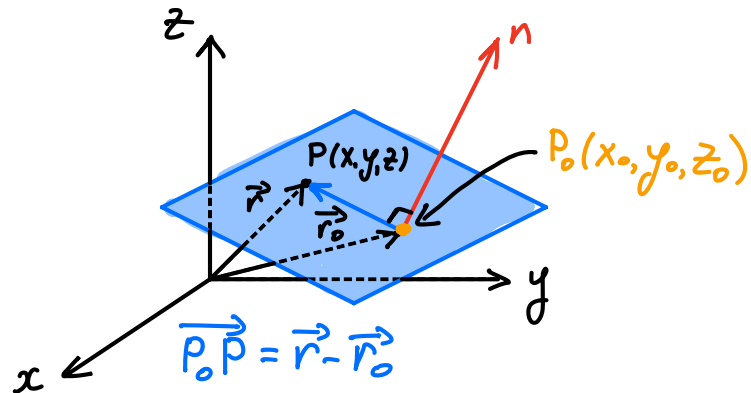
- parametric equations
of the line L
through point $P_0(x_0, y_0, z_0)$
and parallel to the vector $\vec{v} = \langle a, b, c \rangle$

($t =$)

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

- symmetric equations of L

Planes



- A plane is defined by a point P_0 and an orthogonal vector $\vec{n}$ ("the normal vector")

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \quad \text{vector eq. of the plane}$$

If $\vec{n} = \langle a, b, c \rangle$, $\vec{r} = \langle x, y, z \rangle$, $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$, then eq. of the plane is

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

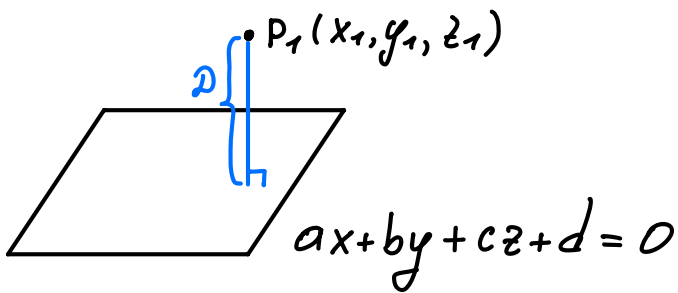
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad \text{- scalar equation of the plane through } P_0(x_0, y_0, z_0) \text{ with normal vector } \vec{n} = \langle a, b, c \rangle$$

Rmk: $ax + by + cz + d = 0$ - linear equation of the plane

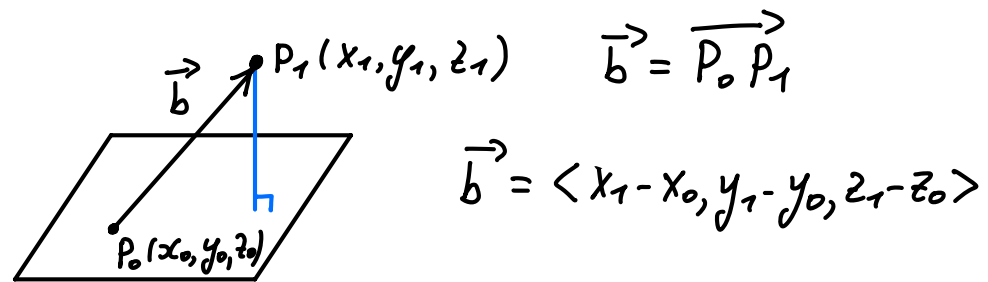
$$d = -(ax_0 + by_0 + cz_0)$$

Distances

Distance From a point to a plane.



Let P_0 be any point in the plane



$$\text{Distance } D = |\text{comp}_{\vec{n}} \overrightarrow{P_0 P_1}| = \frac{|\vec{n} \cdot \vec{b}|}{|\vec{n}|} = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Ex: Find the distance from the point $(1, 1, 1)$
to the plane $x + y + 2z = 0$

Sol:

$$D = \frac{|1 + 1 + 2|}{\sqrt{1 + 1 + 2^2}} = \frac{4}{\sqrt{6}}$$

Vector valued Functions

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

Vector (-valued) Function: domain $\rightarrow$ range
 $\nwarrow$ set of real numbers $\nwarrow$ set of vectors

$$t \mapsto \vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k} \\ = \langle f(t), g(t), h(t) \rangle$$

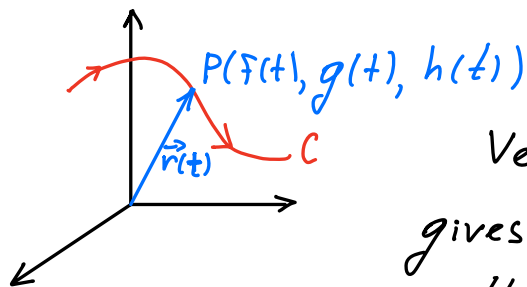
Rmk: domain = values t for which $\vec{r}(t)$ is defined.

Space curves

Set of points (x, y, z) with $t \in I$ interval - "space curve" C .

given functions $\swarrow$
 $x = f(t), y = g(t), z = h(t)$

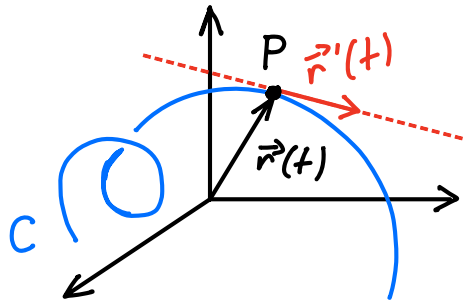
$\swarrow$ parametric equations of C ,
 t -parameter



Vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ gives a "moving" vector, whose tip traces the curve C .

Calculus of space curves

Derivatives: $\vec{r}(t)$ vector function $\leadsto \frac{d\vec{r}}{dt} = \vec{r}'(t) = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$

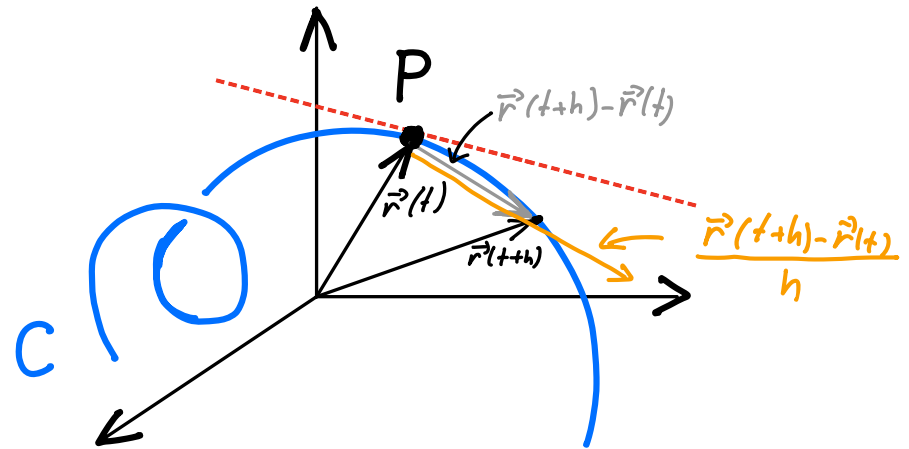


- $\vec{r}'(t)$ - tangent vector to curve C at point P (if $\vec{r}'(t) \neq 0$ and exists)
- tangent line to C at P - line through P , parallel to $\vec{r}'(t)$

• unit tangent vector: $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$

* If $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$

then $\vec{r}'(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$



Second derivative: $\vec{r}'' = (\vec{r}'(t))'$

Ex: $\vec{r}(t) = \langle 2\cos t, \sin t, t \rangle \Rightarrow \vec{r}'(t) = \langle -2\sin t, \cos t, 1 \rangle$
 $\Rightarrow \vec{r}''(t) = \langle -2\cos t, -\sin t, 0 \rangle$

Differentiation rules

$$(\vec{u}(t) + \vec{v}(t))' = \vec{u}'(t) + \vec{v}'(t)$$

$$(c \vec{u}(t))' = c \vec{u}'(t)$$

$$(\vec{u}(f(t)))' = f'(t) \vec{u}'(f(t))$$

↑ chain rule

$$(f(t) \vec{u}(t))' = f'(t) \vec{u}(t) + f(t) \vec{u}'(t)$$

$$(\vec{u}(t) \cdot \vec{v}(t))' = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$$

$$(\vec{u}(t) \times \vec{v}(t))' = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$$

Integrals:

$$\int_a^b \vec{r}(t) dt = \left(\int_a^b f(t) dt \right) \vec{i} + \left(\int_a^b g(t) dt \right) \vec{j} + \left(\int_a^b h(t) dt \right) \vec{k}$$

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

Fund. Thm of Calculus
(in vector setting)

$$\int_a^b \vec{r}(t) dt = \vec{R}(t) \Big|_a^b = \vec{R}(b) - \vec{R}(a)$$

↙ $\vec{R}'(t) = \vec{r}(t)$

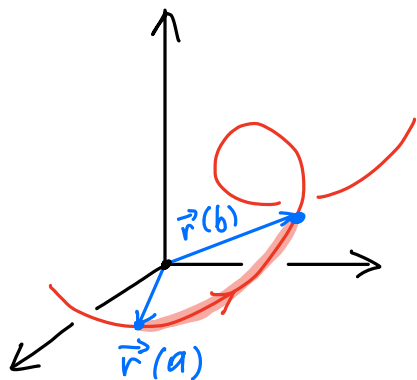
We denote $\vec{R}(t) = \int \vec{r}(t) dt$ (indefinite integral)

Arc length

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2} dt$$

length of the curve

$$= \int_a^b |\vec{r}'(t)| dt$$



$$C: \vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

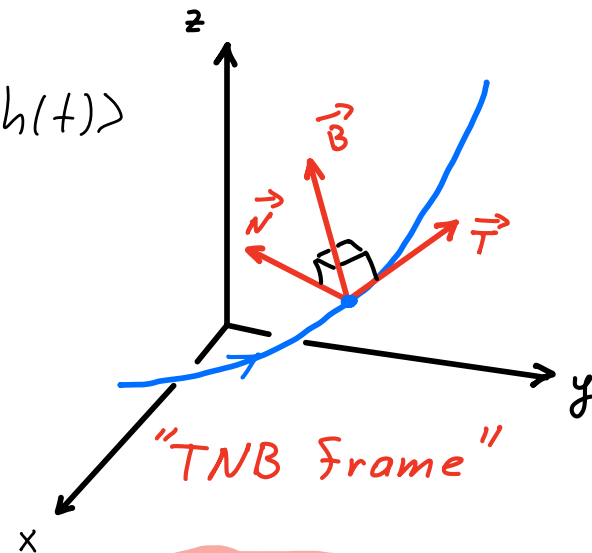
unit tang.

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

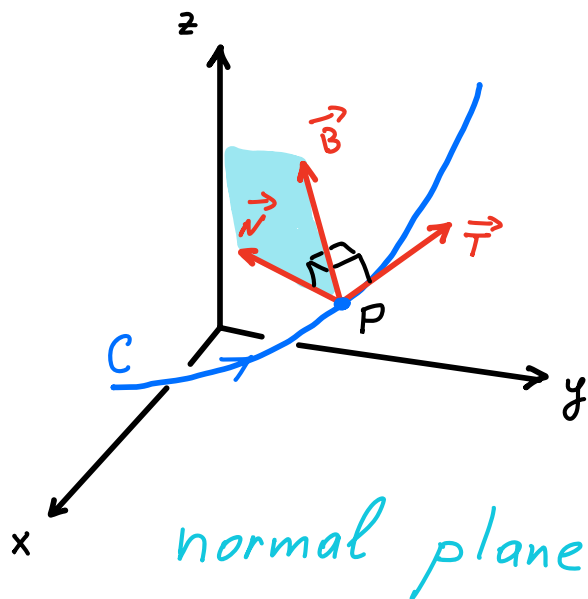
normal

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

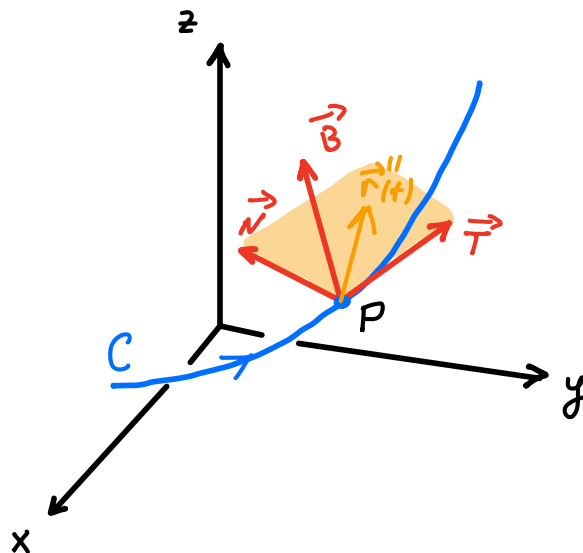
binormal



Rmk: $\vec{T} \times \vec{N} = \vec{B}$
 $\vec{T} = \vec{N} \times \vec{B}$
 $\vec{N} = \vec{B} \times \vec{T}$

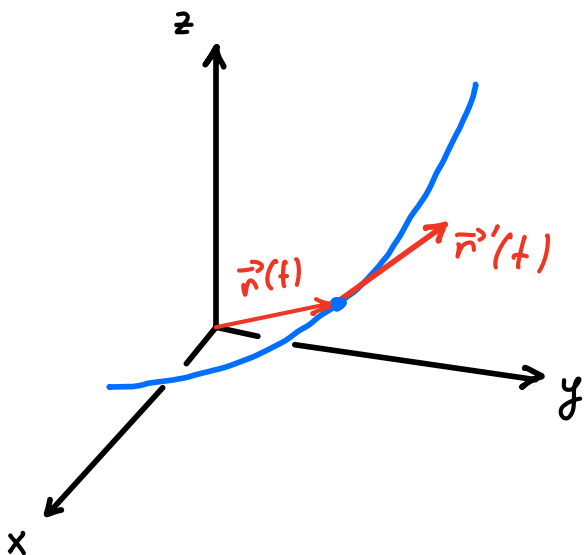


osculating plane



Rmk: $\vec{r}''(t)$ is in the osculating plane.

Motion in space



particle moving through space;
position at time t : $\vec{r}(t)$

$$\vec{r}'(t) = \lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} = \vec{v}(t)$$

displacement vector
per unit time $\uparrow$ velocity vector

speed: $|\vec{v}(t)| = |\vec{r}'(t)| = \frac{ds}{dt} \leftarrow$ arc length fct.

$\uparrow$ rate of change of dist. w.r.t. time

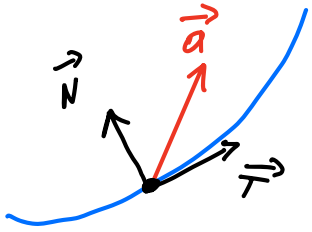
Reminder: $s(t) = \int_a^t |\vec{r}'(u)| du \Rightarrow \frac{ds}{dt} = |\vec{r}'(t)|$

acceleration: $\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$

Rmk: If we know $\vec{a}(t)$ and $\vec{r}(t_0), \vec{v}(t_0)$ (initial position and velocity) then

$$\vec{v}(t) = \int_{t_0}^t \vec{a}(u) du + \vec{v}(t_0), \quad \vec{r}(t) = \int_{t_0}^t \vec{v}(u) du + \vec{r}(t_0)$$

Tangential and normal components of acceleration:



$$\vec{a} = \underbrace{a_T}_{\text{tangential component}} \vec{T} + \underbrace{a_N}_{\text{normal component}} \vec{N}$$

$$a_T = \vec{a} \cdot \vec{T} = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{|\vec{r}'(t)|}$$

$$a_N = \vec{a} \cdot \vec{N} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|}$$

Functions of several variables

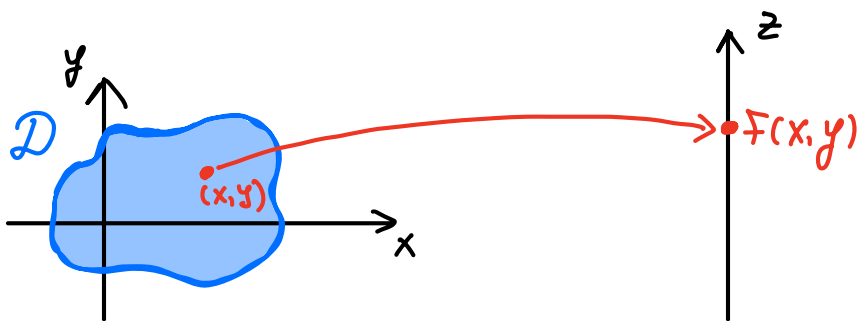
Function of two variables:

$$f: (x, y) \mapsto f(x, y)$$

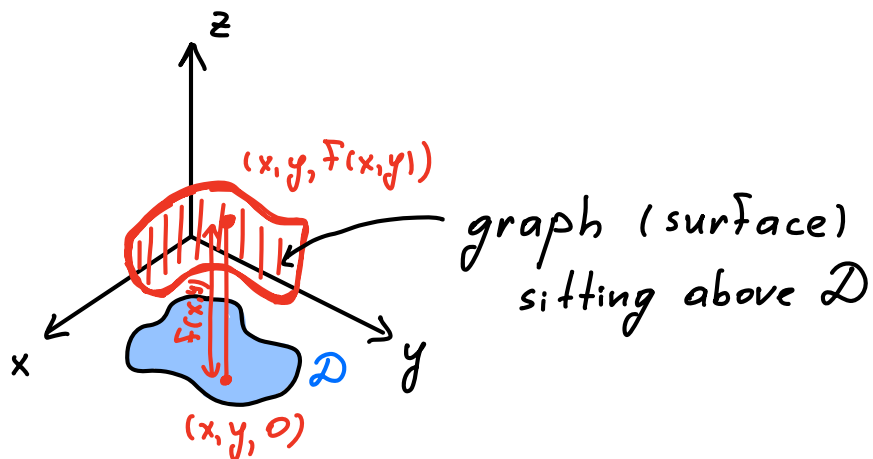
in $\mathcal{D} \subset \mathbb{R}^2$ in $\mathbb{R}$

$\uparrow$ domain

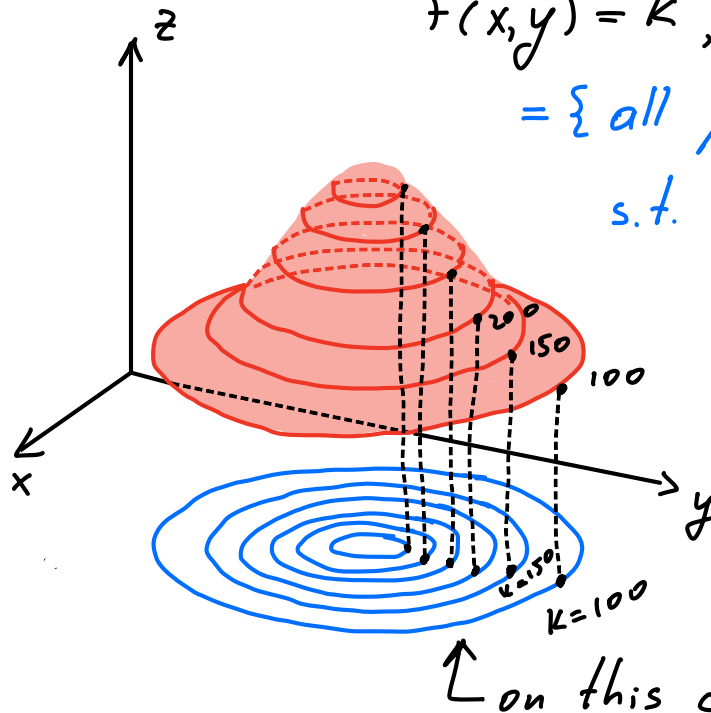
range: set of all values of $f(x, y) \in \mathbb{R}$
domain: all (x, y) s.t. $f(x, y)$ is well-defined.



Graph: Set of points (x, y, z) in $\mathbb{R}^3$
with $(x, y) \in \mathcal{D}$, $z = f(x, y)$



Level curves: curves with eq.
 $f(x, y) = k$, i.e.
 $= \{ \text{all points } (x, y) \text{ s.t. } f(x, y) = k \}$



Functions of 3 variables:

$$f(x, y, z)$$

Partial derivatives

$$f(x, y)$$

partial derivative w.r.t. x at (a, b) :

set $g(x) = f(\underbrace{x}, b)$

Function of single variable

$$F_x(a, b) = g'(a)$$

likewise, partial derivative w.r.t. y at (a, b) :

set $h(y) = F(a, y)$

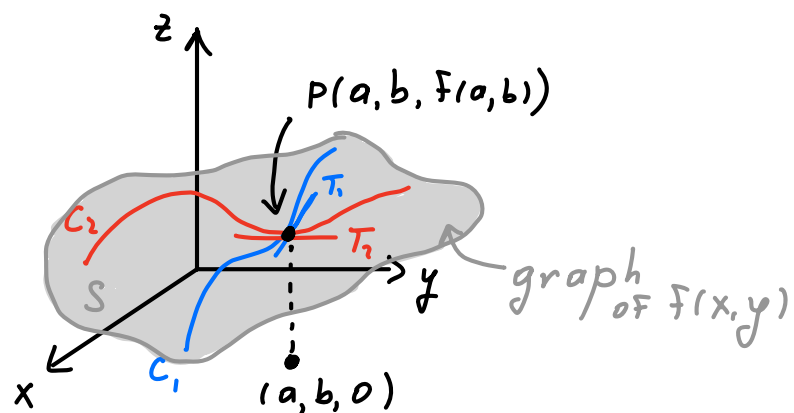
$$f_y(a, b) = h'(b)$$

I.e. to find $f_x(x,y)$, view y as a constant, differentiate w.r.t. x

to find $f_y(x, y)$, view x as a constant, differentiate w.r.t. y

Notation: $\frac{\partial f}{\partial x} = f_x(x, y)$, $\frac{\partial f}{\partial y} = f_y(x, y)$

Interpretations of partial derivatives:



C_1 : graph of $g(x) = f(x, b)$
= intersection of S with plane $y=b$

C_2 : graph of $h(y) = f(a, y)$
= intersection of S with plane $x=a$

$f_x(a, b)$ = slope of tangent line T_1 to C_1 at $P(a, b, f(a, b))$

$f_y(a, b)$ = slope of tangent line T_2 to C_2 at $P(a, b, f(a, b))$

Higher derivatives:

$$\left. \begin{aligned} f_{xx} &= (f_x)_x = \frac{\partial^2 f}{\partial x^2}, & f_{xy} &= (f_x)_y = \frac{\partial^2 f}{\partial x \partial y}, \\ f_{yx} &= (f_y)_x = \frac{\partial^2 f}{\partial y \partial x}, & f_{yy} &= (f_y)_y = \frac{\partial^2 f}{\partial y^2} \end{aligned} \right\} \begin{array}{l} 2^{\text{nd}} \text{ order} \\ \text{derivatives} \end{array}$$

Clairaut's Thm: $f_{xy}(a, b) = f_{yx}(a, b)$

The Chain Rule:

Reminder: $y = f(x)$, $x = g(t)$ then y is indirectly a function of t

$$\text{and } \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = f'(x) \cdot g'(t) = f'(g(t)) \cdot g'(t) \quad [\text{Case 0}]$$

Case 1: $z = f(x, y)$, $x = g(t)$, $y = h(t)$

$$\text{then } \frac{dz}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} \quad \left(\text{or } \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \right)$$

Case 2: $z = f(x, y)$ and $x = g(s, t)$, $y = h(s, t)$

$$\text{then } \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

General case: $u = u(\underbrace{x_1, \dots, x_n}_{n \text{ variables}})$ and for each j : $x_j = x_j(\underbrace{t_1, \dots, t_m}_{m \text{ variables}})$

$$\text{then } \frac{\partial u}{\partial t_i} = \underbrace{\frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i}}_{n \text{ terms}}$$

Implicit differentiation: If $y = f(x)$ is defined implicitly,
by $F(x, y) = 0$

We can find $\frac{dy}{dx}$ via $0 = \frac{d}{dx} F(x, y) = \frac{\partial F}{\partial x} \underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial F}{\partial y} \frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = - \frac{\partial F / \partial x}{\partial F / \partial y} = - \frac{F_x}{F_y}$$

If $z = g(x, y)$ is defined implicitly, by $G(x, y, \underbrace{z}_{=g(x,y)}) = 0$.

We can find $\frac{\partial z}{\partial x}$ via

$$0 = \frac{\partial}{\partial x} G(x, y, z) = \frac{\partial G}{\partial x} \cdot \underbrace{\frac{dx}{dx}}_{=1} + \frac{\partial G}{\partial y} \cdot \underbrace{\frac{dy}{dx}}_{=0} + \frac{\partial G}{\partial z} \cdot \underbrace{\frac{\partial z}{\partial x}}_{\frac{\partial}{\partial x} g(x, y)}$$

$$\Rightarrow \frac{\partial z}{\partial x} = - \frac{\frac{\partial G}{\partial x}}{\frac{\partial G}{\partial z}}$$

Directional derivative:

$$F(x, y)$$

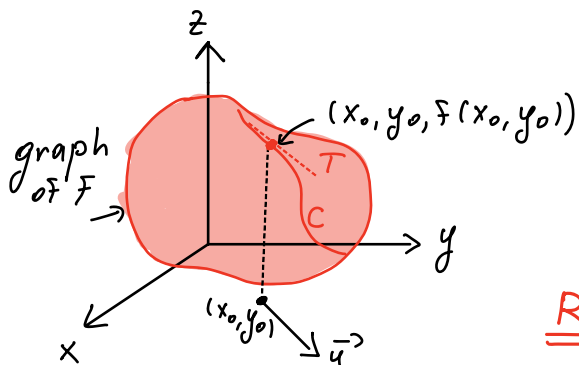
$$\vec{u} = \langle a, b \rangle$$

unit vector

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h\vec{u}, y_0 + h\vec{v}) - f(x_0, y_0)}{h}$$

directional derivative of f at (x_0, y_0) in the direction of $\vec{u}$.

Directional



$$D_{\vec{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

Rmk: $\vec{u} = \langle 1, 0 \rangle \Rightarrow D_{\vec{u}} f = f_x$; $\vec{u} = \langle 0, 1 \rangle \Rightarrow D_{\vec{u}} f = f_y$

Gradient: $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$

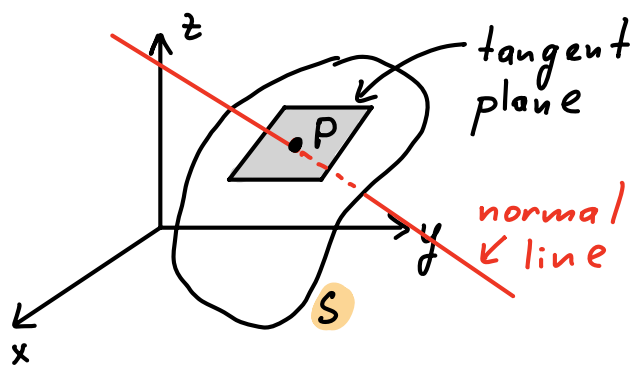
$$D_{\vec{u}} f(x, y) = \nabla f \cdot \vec{u}$$

S: $F(x, y, z) = k$
- level surface

The gradient $\nabla F(x_0, y_0, z_0)$ is orthogonal to the level surface S
the tangent plane to level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

$P(x_0, y_0, z_0)$ point on S

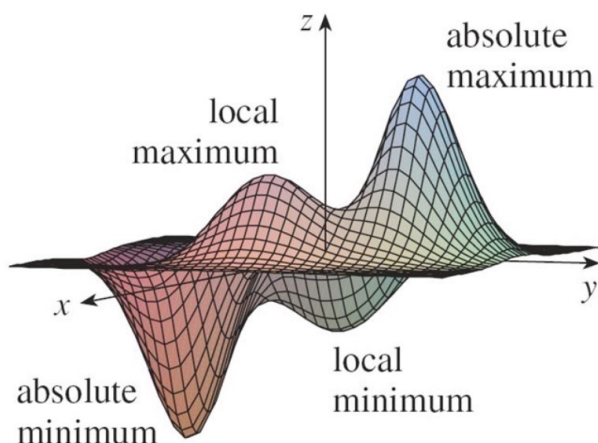


normal line to S at P:

line through P and orthogonal to tangent plane
(i.e. its direction vector is $\nabla F(x_0, y_0, z_0)$)

$$\Rightarrow \frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

$\nabla F(x_0, y_0, z_0)$ - direction of maximal increase of F



Thm: If F has a local maximum or minimum at (a, b) then $F_x(a, b) = 0$ and $F_y(a, b) = 0$.

Rmk: i.e. $\nabla F(a, b) = 0$

Def: If $\nabla F(a, b) = 0$, we call (a, b) a critical point of F .

Second derivatives test

Suppose (a, b) is a critical point of F (i.e. $F_x(a, b) = 0$ and $F_y(a, b) = 0$).

$$\text{Set } D = D(a, b) = F_{xx}(a, b)F_{yy}(a, b) - (F_{xy}(a, b))^2$$

- (a) If $D > 0$ and $F_{xx}(a, b) > 0$, then $F(a, b)$ - local min.;
- (b) If $D > 0$ and $F_{xx}(a, b) < 0$, then $F(a, b)$ - local max.;
- (c) If $D < 0$, then $F(a, b)$ is a saddle point of F .

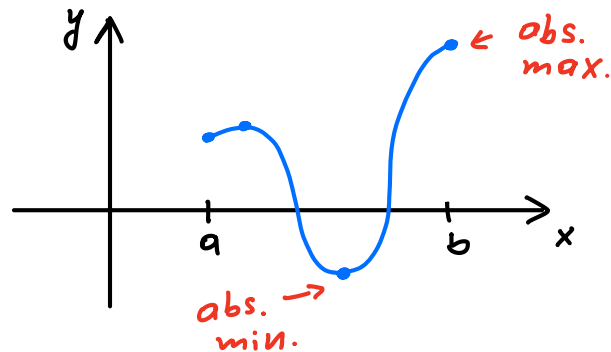
Rmk: If $D = 0$, the test gives no information.

Rmk:

$$D = \begin{vmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{vmatrix} = F_{xx}F_{yy} - (F_{xy})^2$$

Extrema on bounded regions

Recall: $f(x)$ continuous on $[a, b]$, has an absolute max. and an abs. min. on $[a, b]$.

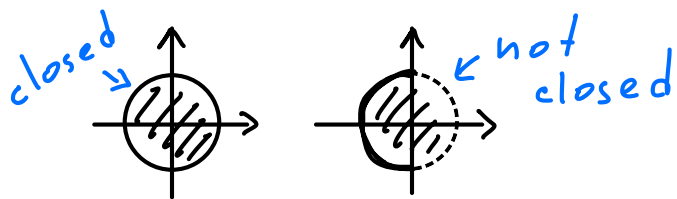


- We find them by evaluating f on critical points **AND** on endpoints.

$\mathcal{D}$ -closed set in $\mathbb{R}^2$

(i.e. contains boundary pts);

bounded set (contained in some disk)



To Find abs. max/min values of f on a closed, bounded $\mathcal{D}$:

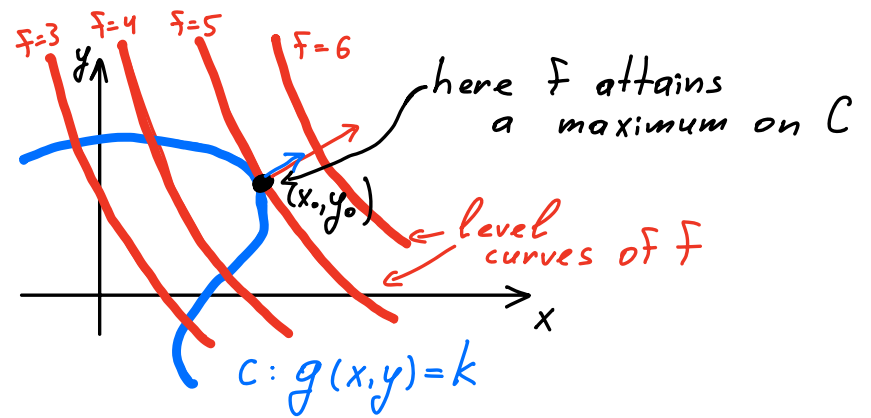
- ① Find values of f at critical points in $\mathcal{D}$;
- ② Find extreme values on the boundary of $\mathcal{D}$;
- ③ largest of ①, ② - abs. max.
smallest of ①, ② - abs. min.

f , continuous on a closed, bounded $\mathcal{D} \subset \mathbb{R}^2$,
attains an abs. max. value $f(x_1, y_1)$
and an abs. min. value $f(x_2, y_2)$ at some
points in $\mathcal{D}$.

Lagrange multipliers

Problem:

Find extreme values of $f(x,y)$
subject to constraint $g(x,y)=k$



level curves $g(x,y)=k$ and $f(x,y)=6$ touch at (x_0, y_0)

$$\Rightarrow \nabla f(x_0, y_0) = \underbrace{\lambda}_{\substack{\uparrow \text{a number, "Lagrange multiplier"}}} \nabla g(x_0, y_0)$$

$\uparrow$ a number, "Lagrange multiplier"

Method of Lagrange multipliers

To find maximum & minimum values of $f(x,y)$

subject to constrain $g(x,y)=k$:

① Find all values of x, y, λ such that

$$\bullet \nabla f(x,y) = \lambda \nabla g(x,y)$$

$$\bullet g(x,y) = k$$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \end{cases}$$

② evaluate f at all points found in (a)

maximum = largest value of f

minimum = smallest

Method of Lagrange multipliers (TWO CONSTRAINTS)

To find maximum & minimum values of $f(x, y, z)$
subject to constraints $g(x, y, z) = k$ and $h(x, y, z) = c$

① Find all values of x, y, z, λ, μ such that

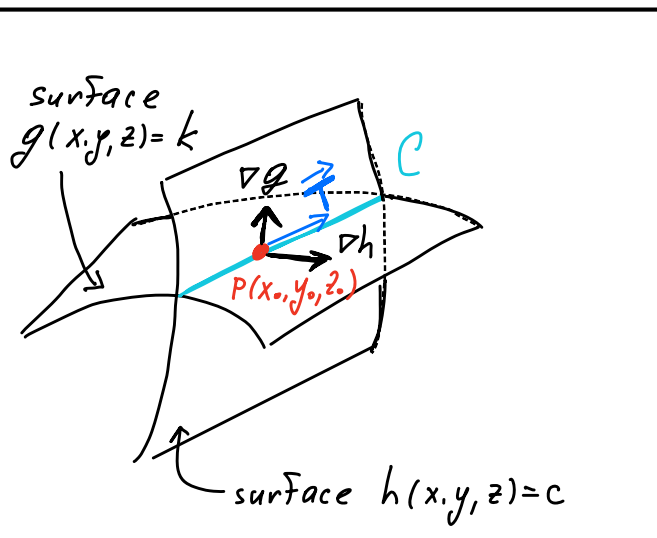
- $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z) \leftarrow$
- $g(x, y, z) = k$
- $h(x, y, z) = c$

$$\begin{aligned} f_x &= \lambda g_x + \mu h_x \\ f_y &= \lambda g_y + \mu h_y \\ f_z &= \lambda g_z + \mu h_z \end{aligned}$$

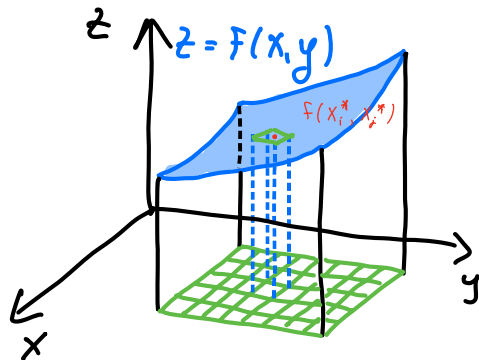
② evaluate f at all points found in (a)

maximum = largest value of f

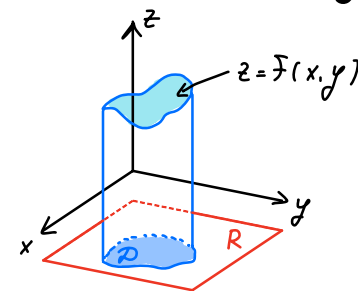
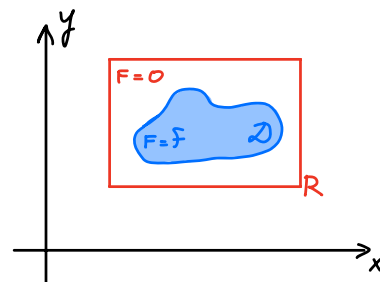
minimum = smallest



$\iint_R F(x,y) dA = \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \sum_{i,j} f(x_i^*, y_j^*) \underbrace{\Delta A}_{\Delta x \cdot \Delta y} \stackrel{\text{for } F \geq 0}{=} \text{volume of a solid that lies above the rectangle (general region) and below the surface } z = f(x,y).$

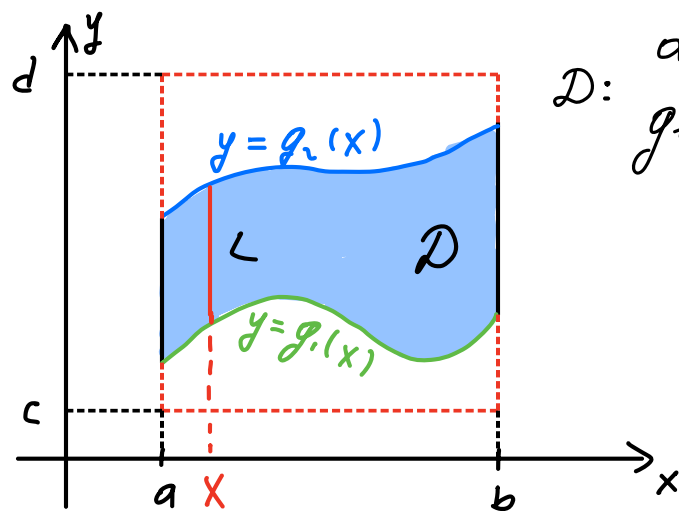


(x_i^*, y_j^*) - sample point in $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$



Fubini's thm: if $f(x,y)$ is continuous

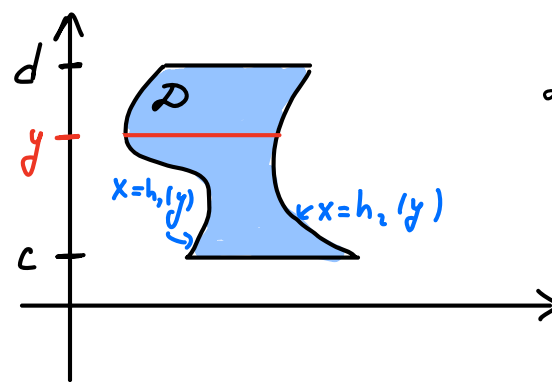
$$\iint_R f(x,y) dA = \int_a^b \int_c^d f(x,y) dy dx = \int_c^d \int_a^b f(x,y) dx dy$$



$D: a \leq x \leq b$
 $g_1(x) \leq y \leq g_2(x)$

$$\iint_D f(x,y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y) dy dx$$

region of type **I**

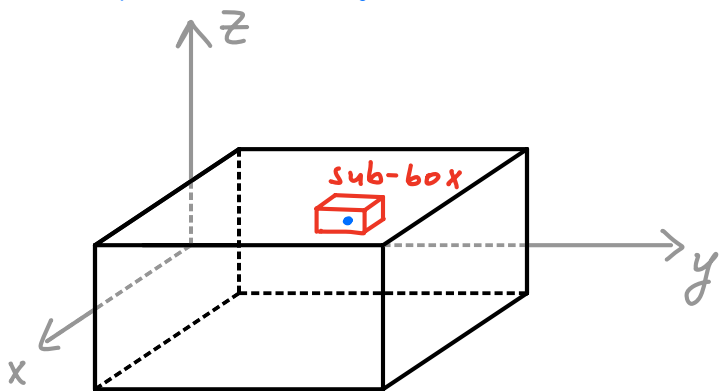


$D: c \leq y \leq d$
 $h_1(y) \leq x \leq h_2(y)$

region of type **II**

$$\iint_D f(x,y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y) dx dy$$

Triple integrals



$$B = \{ (x, y, z) \mid \begin{array}{l} a \leq x \leq b \\ c \leq y \leq d \\ r \leq z \leq s \end{array} \} \text{ rectangular box}$$

Triple integral: $f(x, y, z)$ - continuous fct.

$$\iiint_B f(x, y, z) dV =$$

$$\lim_{l, m, n} \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n f(\underbrace{x_{ijk}^*, y_{ijk}^*, z_{ijk}^*}_{\substack{\text{sample point in} \\ (ijk)\text{-th sub-box}}}) \Delta V$$

Fubini's thm: if $f(x, y, z)$ continuous

$$\iiint_B f(x, y, z) dV = \int_r^s \int_c^d \int_a^b f(x, y, z) dx dy dz$$

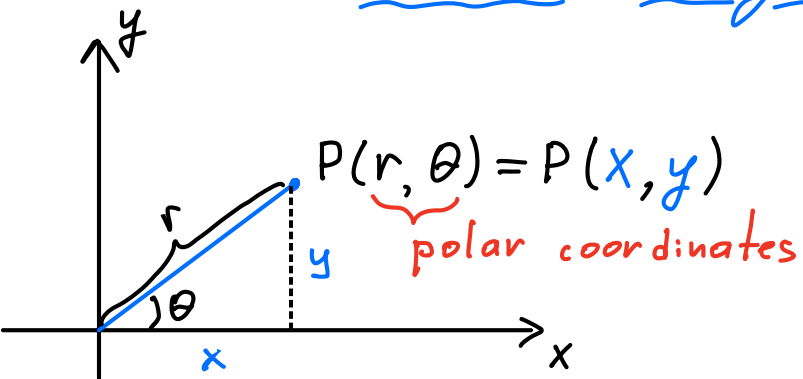
Rmk: Can be written in 5 other orders, e.g.

$$\int_c^d \int_a^b \int_r^s f(x, y, z) dz dx dy$$

Application:

$$\iiint_E 1 dV = \text{Volume of the solid } E$$

Double integrals in polar coordinates



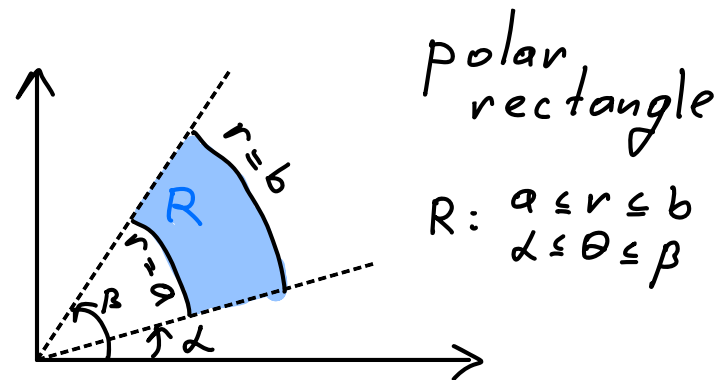
$$x = r \cos \theta$$

$$y = r \sin \theta$$

don't forget
this factor r !

$$\iint_R F(x, y) dA = \int_{\alpha}^{\beta} \int_a^b F(r \cos \theta, r \sin \theta) \underline{r} dr d\theta$$

Assume: $0 \leq a \leq b$, $0 \leq \beta - \alpha \leq 2\pi$

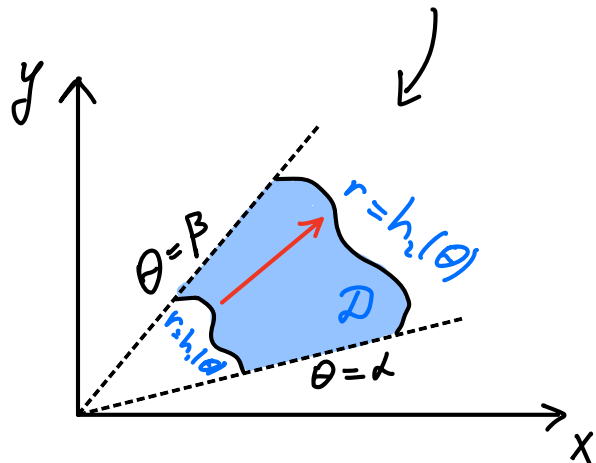


polar
rectangle

$$R: a \leq r \leq b$$

$$\alpha \leq \theta \leq \beta$$

For D like that

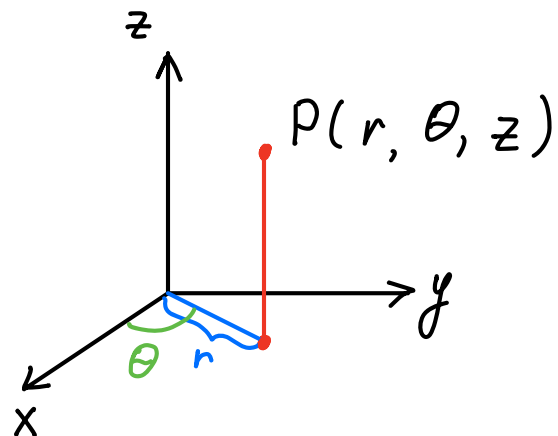


$$D = \{ (r, \theta) \mid \alpha \leq \theta \leq \beta$$

$$h_1(\theta) \leq r \leq h_2(\theta) \}$$

$$\iint_D F(x, y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} F(r \cos \theta, r \sin \theta) r dr d\theta$$

Triple integrals in cylindrical coordinates



r, θ, z - cylindrical coordinates
of a point P
polar coord.
of proj. of P
onto xy -plane

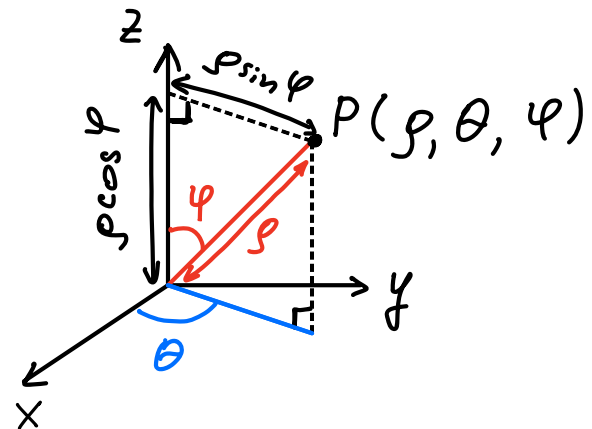
$$\left. \begin{array}{l} x = r \cos \theta \\ y = r \sin \theta \end{array} \right\} \Rightarrow x^2 + y^2 = r^2$$
$$z = z$$

For a solid E :

$$\left\{ \begin{array}{l} \alpha \leq \theta \leq \beta \\ h_1(\theta) \leq r \leq h_2(\theta) \\ g_1(r, \theta) \leq z \leq g_2(r, \theta) \end{array} \right\}$$

$$\iiint_E f(x, y, z) dv = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{g_1(r, \theta)}^{g_2(r, \theta)} f(r \cos \theta, r \sin \theta, z) \underline{r} dz dr d\theta$$

Triple integrals in spherical coordinates



$$\rho = |OP|$$

θ - same as in cylindrical

φ - angle between $\vec{OP}$ and z -axis

$$\rho \geq 0, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \varphi \leq \underline{\underline{\pi}}$$

$$z = \rho \cos \varphi$$

$$x = \rho \sin \varphi \cos \theta$$

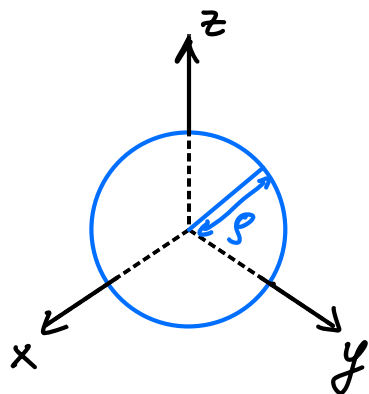
$$y = \rho \sin \varphi \sin \theta$$

$$r = \rho \sin \varphi$$

↑ r of cylindrical

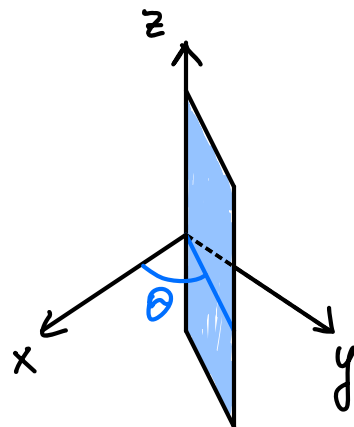
Rmk: $\rho^2 = x^2 + y^2 + z^2$

$$\rho = \text{const}$$



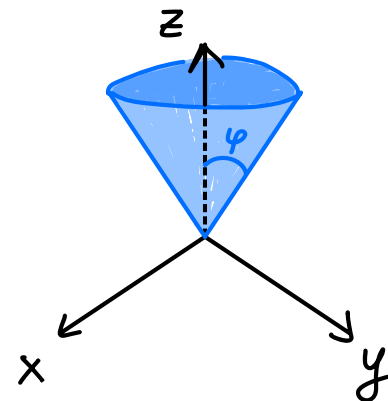
A sphere

$$\theta = \text{const}$$



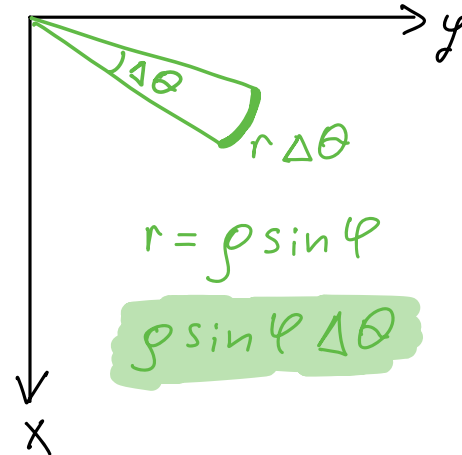
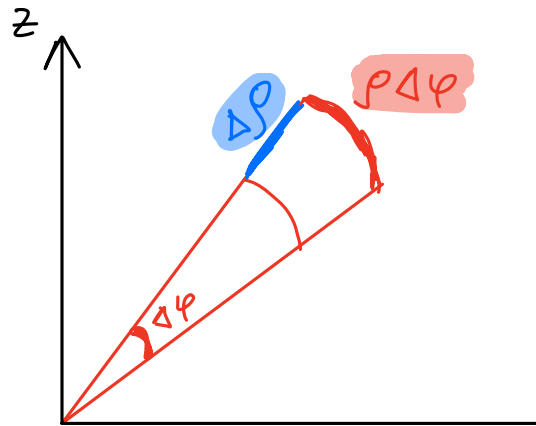
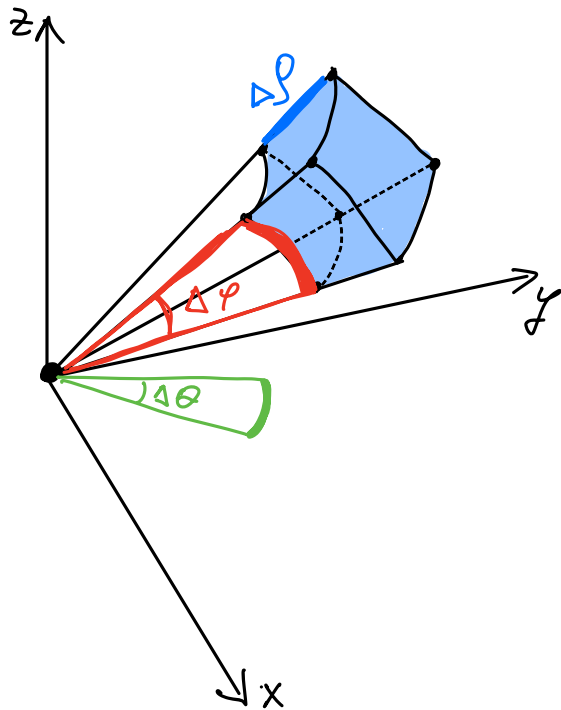
A half-plane

$$\varphi = \text{const}$$



A cone

$$E = \{ (\rho, \theta, \varphi) \mid \left. \begin{array}{l} c \leq \varphi \leq d \\ \alpha \leq \theta \leq \beta \\ a \leq \rho \leq b \end{array} \right\} \quad \text{- spherical wedge}$$

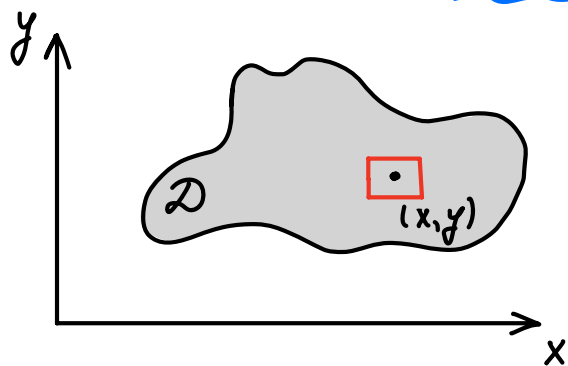


$$\Delta V = \rho^2 \sin \varphi \Delta \varphi \Delta \theta \Delta \rho$$

$$\iiint_E F(x, y, z) dV = \int_c^d \int_\alpha^\beta \int_a^b F(\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi) \rho^2 \sin \varphi d\rho d\theta d\varphi$$

Rmk: Can extend this to more general spherical regions
 $E: c \leq \varphi \leq d, \alpha \leq \theta \leq \beta, g_1(\theta, \varphi) \leq \rho \leq g_2(\theta, \varphi)$

Mass, Center of Mass, Moments



A lamina (plate),

• density $\rho(x, y) = \lim \frac{\Delta m}{\Delta A}$
(in units of mass per unit area)

mass in a tiny rectangle about (x, y)

area of the rectangle

• total mass

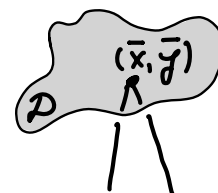
$$m = \lim_{k, \ell \rightarrow \infty} \sum_{i=1}^k \sum_{j=1}^{\ell} \underbrace{\rho(x_{ij}^*, y_{ij}^*) \Delta A}_{\approx \text{mass in } ij\text{-th rectangle}} = \iint_D \rho(x, y) dA$$

Moment of a lamina about x-axis: $M_x = \iint_D y \rho(x, y) dA$

Similarly: $M_y = \iint_D x \rho(x, y) dA$

Center of mass $(\bar{x}, \bar{y})$: $m \bar{x} = M_y$, $m \bar{y} = M_x$.

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_D x \rho(x, y) dA}{\iint_D \rho(x, y) dA} \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{\iint_D y \rho(x, y) dA}{\iint_D \rho(x, y) dA}$$



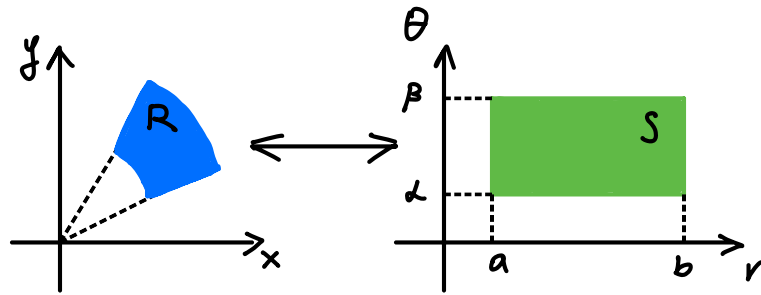
lamina balances when supported at the center of mass

Change of variables in multiple integrals

in 1D integrals: $\int_a^b F(x) dx = \int_c^d F(g(u)) g'(u) du = \int_c^d F(x(u)) \frac{dx}{du} du$

↖ substitution $x = g(u)$
with $a = g(c)$
 $b = g(d)$

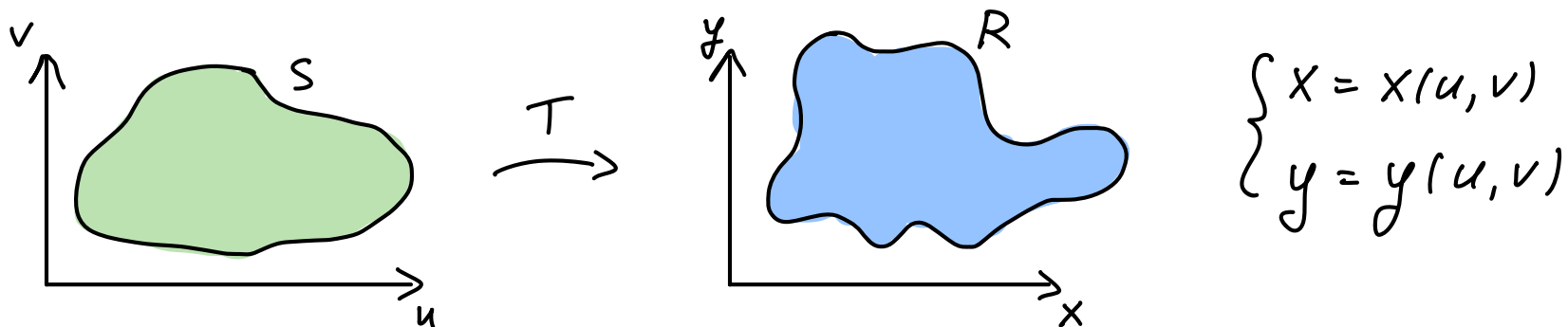
in 2D integrals:
(example)



$$\iint_R F(x, y) dA = \iint_S F(r \cos \theta, r \sin \theta) \underline{r} dr d\theta$$

↖ region in $r\theta$ -plane corresponding
to R (region in xy -plane)

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}$$



Jacobian of T :

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$

Change of variables in a double integral:

Suppose T maps S (in uv -plane) to R (in xy -plane)

and is differentiable and 1-to-1. Suppose Jacobian $\neq 0$ everywhere inside S .

Then

$$\iint_R f(x, y) dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

Ex: (Polar coordinates)

$$T: (r, \theta) \rightarrow (x, y)$$

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}$$

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \underbrace{\frac{\partial x}{\partial r}}_{\cos \theta} \underbrace{\frac{\partial y}{\partial \theta}}_{r \cos \theta} - \underbrace{\frac{\partial x}{\partial \theta}}_{-r \sin \theta} \underbrace{\frac{\partial y}{\partial r}}_{\sin \theta} = \underline{r} \Rightarrow$$

$$\begin{aligned} \iint_R f(x, y) dA \\ = \iint_S f(r \cos \theta, r \sin \theta) \underline{r} dr d\theta \end{aligned}$$

Change variables in triple integral:

$$\iiint_R F(x, y, z) dV = \iiint_S F(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du dv dw,$$

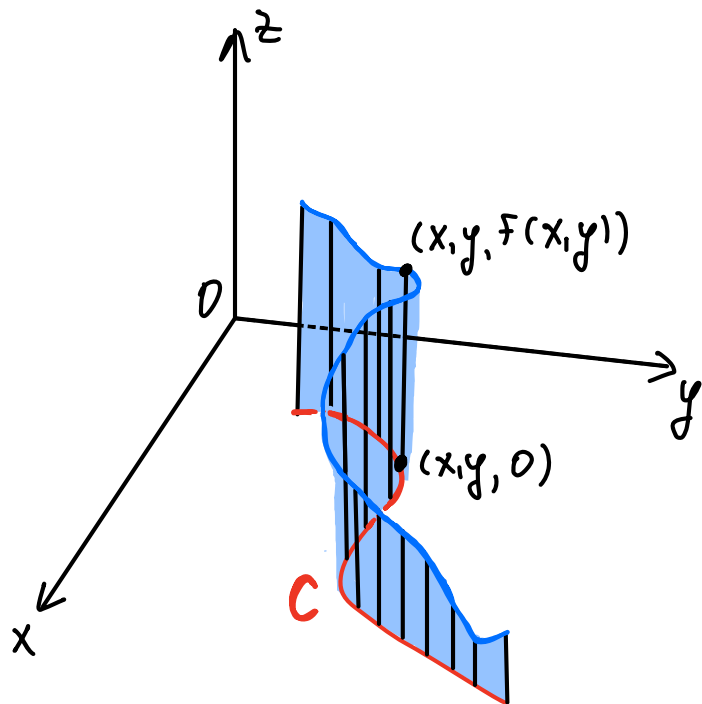
$$\text{where } \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Example: $T: (\rho, \theta, \varphi) \mapsto (x, y, z)$

$$\begin{aligned} x &= \rho \sin \varphi \cos \theta \\ y &= \rho \sin \varphi \sin \theta \\ z &= \rho \cos \varphi \end{aligned} \quad \frac{\partial(x, y, z)}{\partial(\rho, \theta, \varphi)} = \begin{vmatrix} \sin \varphi \cos \theta & -\rho \sin \varphi \sin \theta & \rho \cos \varphi \cos \theta \\ \sin \varphi \sin \theta & \rho \sin \varphi \cos \theta & \rho \cos \varphi \sin \theta \\ \cos \varphi & 0 & -\rho \sin \varphi \end{vmatrix}$$
$$= -\rho^2 \sin \varphi \leq 0 \quad \Rightarrow$$

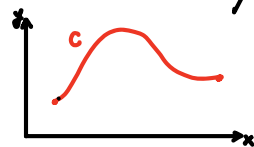
$$\iiint_E F(x, y, z) dV = \iiint_S F(\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi) \rho^2 \sin \varphi d\rho d\theta d\varphi$$

Line integrals of functions



C-plane curve:

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \quad a \leq t \leq b$$



Line (curve) integral of $f(x, y)$ along C :

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Rmk: If $f(x, y) \geq 0$ then $\int_C f(x, y) ds$ represents the area of one side of the "fence"

Line integrals w.r.t. x, y :

$$\int_C f(x, y) d\underline{x} = \int_a^b f(x(t), y(t)) \underline{x'(t)} dt$$

$$\int_C f(x, y) d\underline{y} = \int_a^b f(x(t), y(t)) \underline{y'(t)} dt$$

If C is the form of a (thin) wire with linear density $\rho(x, y)$, then $m = \int_C \rho(x, y) ds$ - mass

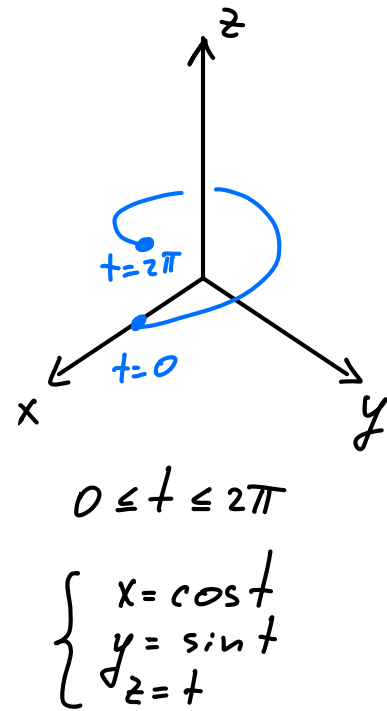
$$\bar{x} = \frac{1}{m} \int_C x \rho(x, y) ds$$

$$\bar{y} = \frac{1}{m} \int_C y \rho(x, y) ds$$

$(\bar{x}, \bar{y})$ - center of mass.

• In space: $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$

$$\int_C F(x, y, z) ds = \int_a^b F(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

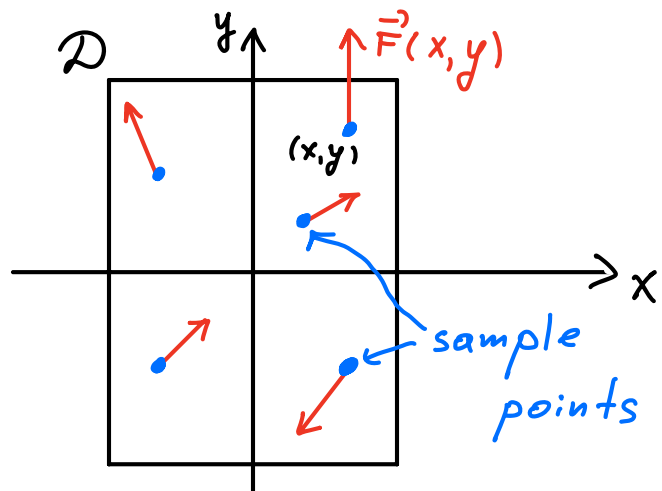


Rmk: $\int_C P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz$

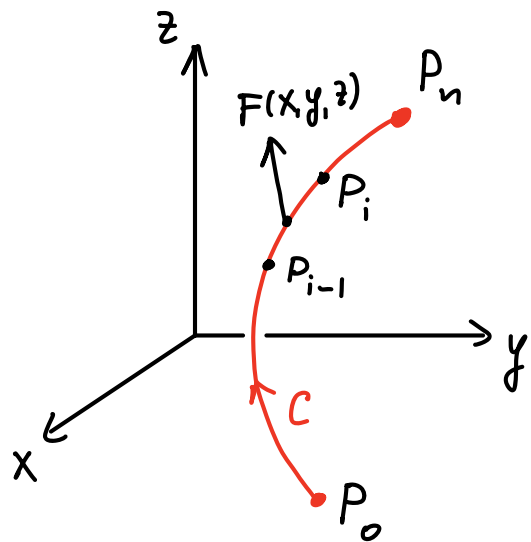
$$= \int_a^b (P(x(t), y(t), z(t)) x'(t) + Q(x(t), y(t), z(t)) y'(t) + R(x(t), y(t), z(t)) z'(t)) dt$$

$$\parallel \int_C P(x, y, z) \underline{dx} = \int_a^b P(x(t), y(t), z(t)) \underline{x'(t)} dt \parallel$$

Vector Field:



A vector field on $\mathbb{R}^2$ is a function $\vec{F}(x, y)$ assigning to each point $(x, y) \in \mathcal{D}$ a 2D vector $\vec{F}(x, y)$ region in $\mathbb{R}^2$



C-curve, $\vec{F}(x, y, z)$ - Force Field
Work W done by force $\vec{F}$ while moving a particle along curve C :

$$W = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

Line integral of a vector field $\vec{F}$ along a curve C given by $\vec{r}(t)$, $a \leq t \leq b$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \underbrace{\vec{F}(\vec{r}(t))}_{\vec{F}(x(t), y(t), z(t))} \cdot \vec{r}'(t) dt$$

Fundamental theorem of line integrals

Recall: a) Fund. thm. of Calculus :

$$\int_a^b F'(x) dx = F(b) - F(a)$$

b) gradient vector field $\vec{F}(x, y) = \nabla f(x, y)$

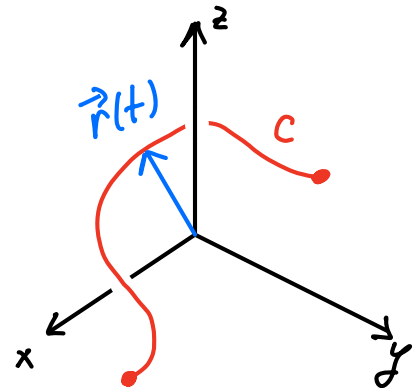
↳ "potential function"
of $\vec{F}$

Rmk: A vector field with a potential is called "conservative".

Fundamental THM of line integrals:

Let C be a curve (on a plane or in space) given by $\vec{r}(t)$, $a \leq t \leq b$. Then

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$



Conservative vector field

A vector field $\vec{F}$ with a potential is called "conservative".

i.e. there exists a function f s.t.: $\vec{F}(x,y) = \nabla f(x,y)$

↳ "potential function"
of $\vec{F}$

$\mathbb{F}^n$ -vector field. The following are equivalent:

(a) $\vec{F}$ is conservative,

i.e. $\vec{F} = \nabla \phi$

For some function f .

(b)

A diagram showing a region in the xy -plane bounded by two curves, C_1 and C_2 , connecting points A and B . C_1 is the upper curve and C_2 is the lower curve. Arrows on both curves point from A to B , indicating a counter-clockwise orientation of the boundary.

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$(c) \int_C \vec{F} \cdot d\vec{r} = 0$$

For any closed curve

Let $\vec{F}(x,y)$ be a vector field in a simply-connected region D

↗ single-piece, without holes

$$\vec{F}(x, y) = P(x, y)\vec{i} + Q(x, y)\vec{j}$$

THEN $\vec{F}(x, y)$ is conservative iff

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

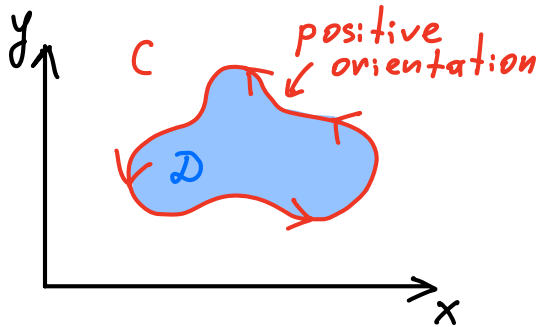
Green's theorem

Simple, closed curve
($\nrightarrow$ not crossing itself)

with positive orientation (counterclockwise)

($\nrightarrow$ region is always on the left
as we go around C)

bounding the region D .

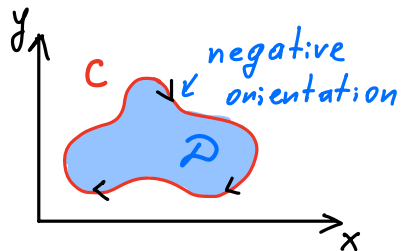


Green's THM:

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$\nrightarrow$ positively oriented

Rmk: For a negatively oriented curve C :



$$\int_C P dx + Q dy = - \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Curl: For $\vec{F}(x,y,z) = \langle P, Q, R \rangle$ - vector field, If $\vec{F}$ is conservative,
 $\text{curl } \vec{F} = \nabla \times \vec{F} = \langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle$
mnemonic Then $\text{curl } \vec{F} = 0$.
 Moreover, for $\vec{F}$ on entire $\mathbb{R}^3$,
 if $\text{curl } \vec{F} = 0$ then $\vec{F}$ is conservative.

 $\text{curl}(\nabla f) = \vec{0}$ for any function f .

Divergence: $\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$
mnemonic
 $\langle P, Q, R \rangle$

 $\text{div}(\text{curl } \vec{F}) = 0$
 for any vector-field $\vec{F}$.

For $\vec{F} = \langle P(x,y), Q(x,y), 0 \rangle$ one has

$$\text{curl } \vec{F} = \vec{k} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

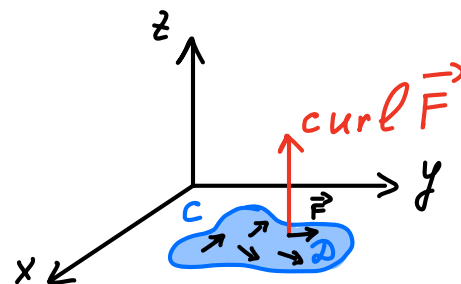
$$\Rightarrow \vec{k} \cdot \text{curl } \vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$\text{div } \nabla f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

- Laplace operator.

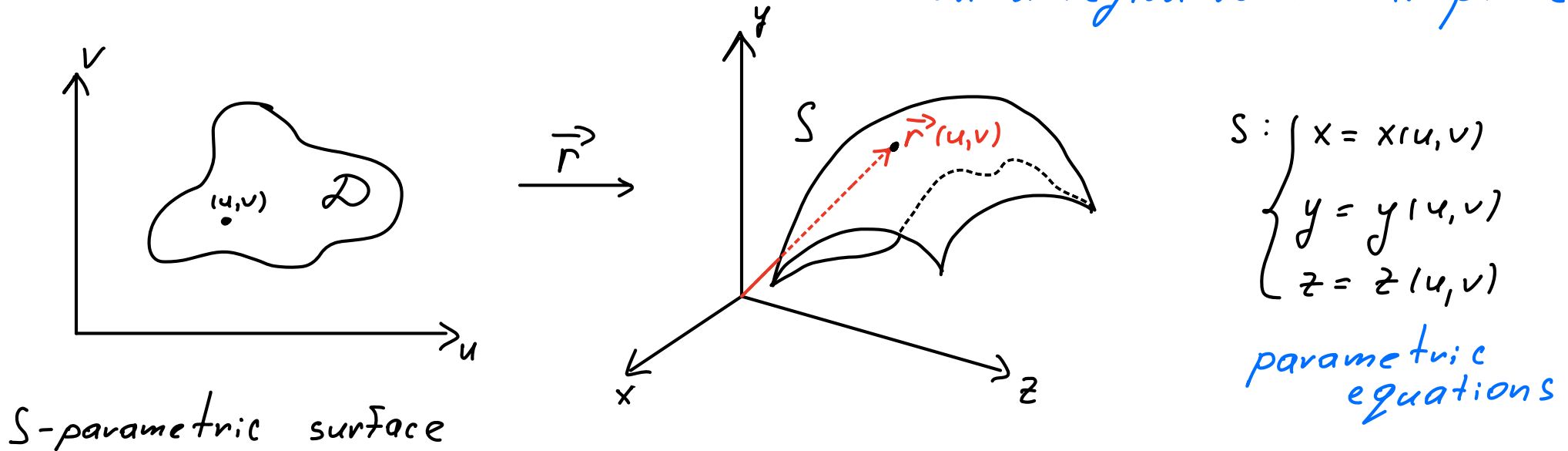
Green's theorem:

$$\Rightarrow \oint_C \vec{F} \cdot d\vec{r} = \int_D \underbrace{(\text{curl } \vec{F}) \cdot \vec{k}}_{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}} dA$$

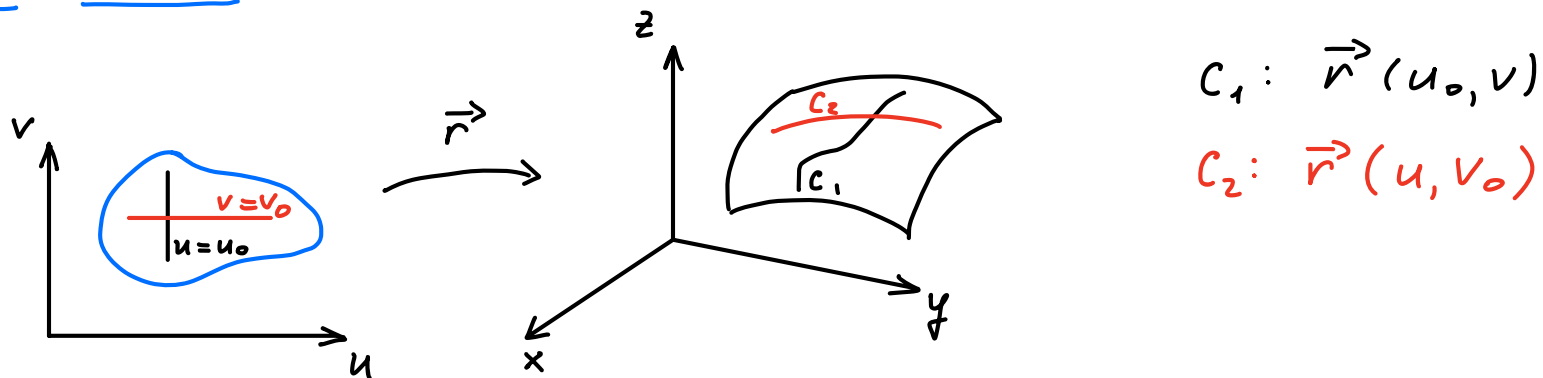


Parametric surfaces

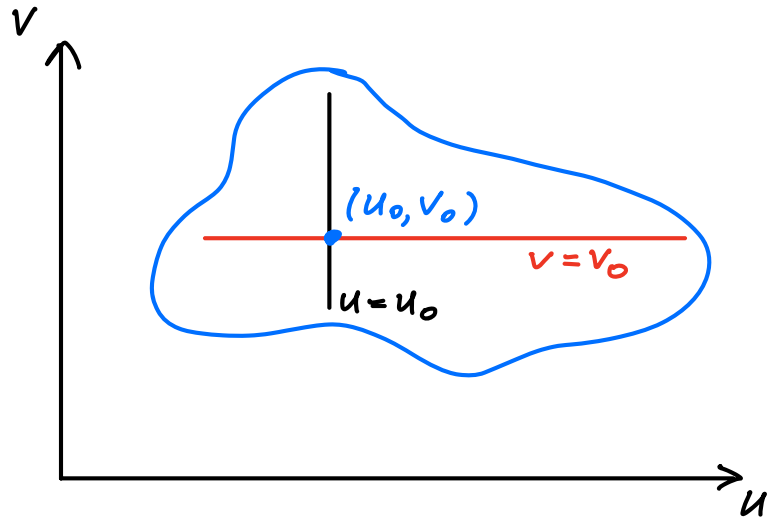
A surface can be described by $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$
vector-valued fct. of u, v
on a region $\mathcal{D}$ in uv -plane



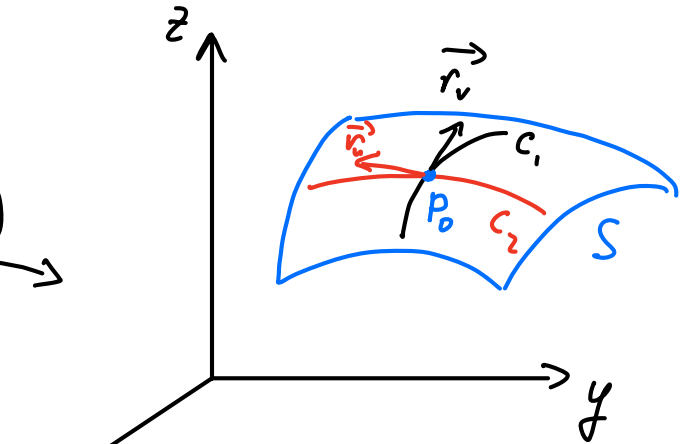
Grid curves:



Tangent plane to a parametric surface



$$\vec{r}(u, v)$$



$$C_1: \vec{r}(u_0, v)$$

$$C_2: \vec{r}(u, v_0)$$

$$\vec{OP}_0 = \vec{r}(u_0, v_0)$$

$$\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$$

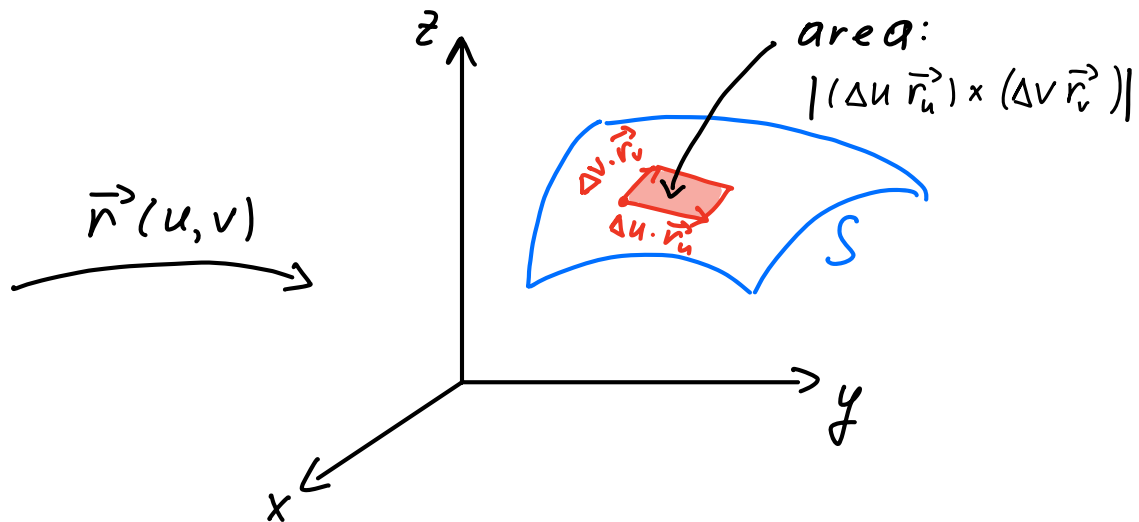
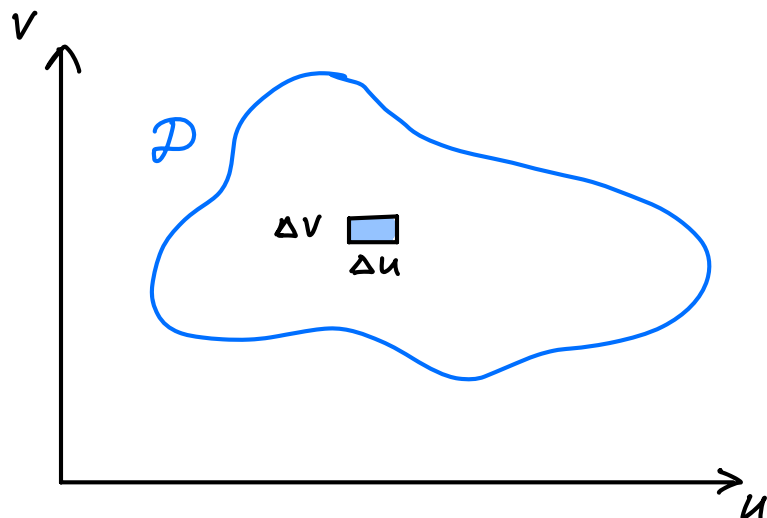
$$\vec{r}_v = \left\langle \frac{\partial x}{\partial v}(u_0, v_0), \frac{\partial y}{\partial v}(u_0, v_0), \frac{\partial z}{\partial v}(u_0, v_0) \right\rangle - \text{tangent vector for } C_1 \text{ at } P_0$$

$$\vec{r}_u = \left\langle \frac{\partial x}{\partial u}(u_0, v_0), \frac{\partial y}{\partial u}(u_0, v_0), \frac{\partial z}{\partial u}(u_0, v_0) \right\rangle - \text{tangent vector for } C_2 \text{ at } P_0$$

Assume $\vec{r}_u \times \vec{r}_v \neq \vec{0}$.
(then S is "smooth")

Then the tangent plane at P_0
has normal vector $\vec{r}_u \times \vec{r}_v$.

Surface area:



Surface area of S :
(a smooth parametric surface)

$$\text{Area}(S) = \iint_{\mathcal{D}} |\vec{r}_u \times \vec{r}_v| dA,$$

where $\vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle,$

$$\vec{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle.$$

Rmk: We assume that S is covered once when (u, v) ranges throughout $\mathcal{D}$.

Surface area of the graph of a function:

$$S: z = f(x, y)$$

$$\text{i.e. } \vec{r}(x, y) = \langle x, y, f(x, y) \rangle$$

$$\Rightarrow \begin{cases} \vec{r}_x = \langle 1, 0, f_x(x, y) \rangle \\ \vec{r}_y = \langle 0, 1, f_y(x, y) \rangle \end{cases}$$

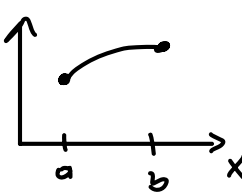
$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle$$

$$\text{and } |\vec{r}_x \times \vec{r}_y| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

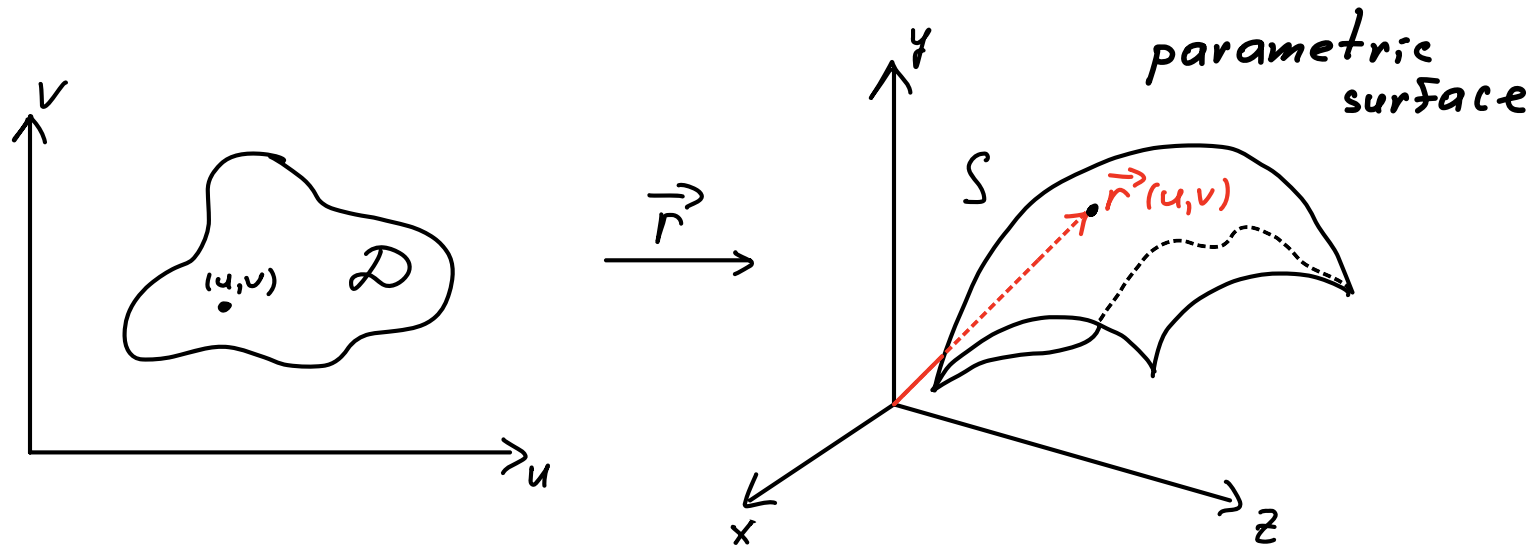
$$\Rightarrow \text{Area}(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA$$

Rmk: Recall the arc length formula:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$



Surface integrals:



Surface integral of a function:

$$\iint_S f(x, y, z) dS = \iint_D f(\vec{r}(u, v)) \underbrace{|\vec{r}_u \times \vec{r}_v|}_{dS} \overbrace{dA}^{du dv}$$

Flux integral

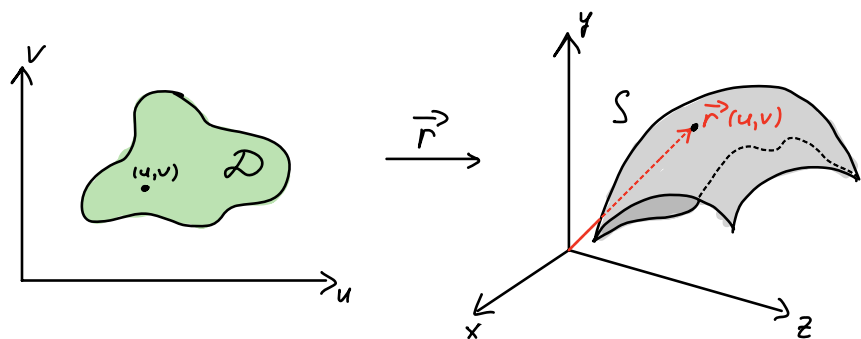
Flux integral is a surface integral of a vector field:

$$\iint_S \vec{F} \cdot d\vec{S} \stackrel{\text{def}}{=} \iint_S \vec{F} \cdot \vec{n} dS = \iint_{\mathcal{D}} \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \underbrace{dA}_{du dv}$$

$\vec{F}$ vector field $\vec{F}(x,y,z)$
 $\vec{n}$ unit normal vector field for S
 $\mathcal{D}$ in uv -plane

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} - \text{defines an orientation}$$

(default orientation induced by parametrization)



Rmk: If S is a graph of some function $g(x,y)$, then

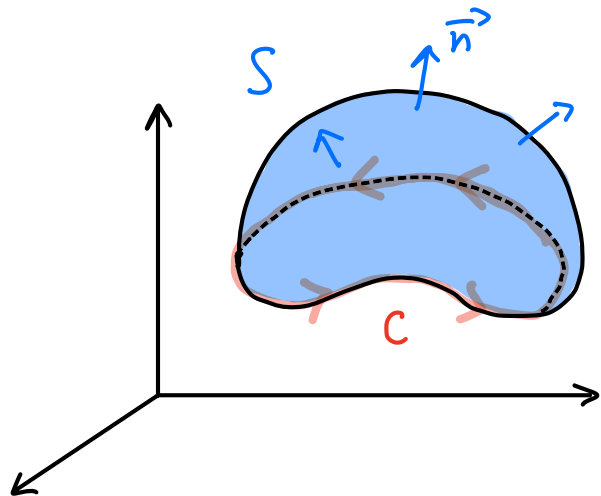
$$\vec{r}(x,y) = \langle x, y, g(x,y) \rangle$$

$$\vec{r}_x \times \vec{r}_y = \langle -g_x, -g_y, 1 \rangle$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{\mathcal{D}} (-Pg_x - Qg_y + R) dA$$

assuming upward orientation of S

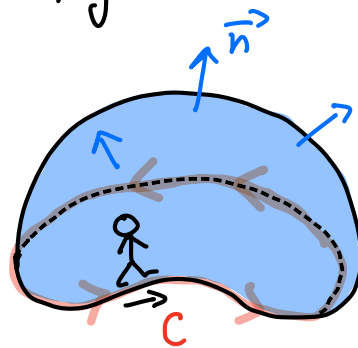
Stoke's theorem



S - oriented surface (choice of $\vec{n}$),
bounded by a closed curve C .

C has an "induced orientation":

- walking along C with your head in the direction of $\vec{n}$, you should see S on your left.



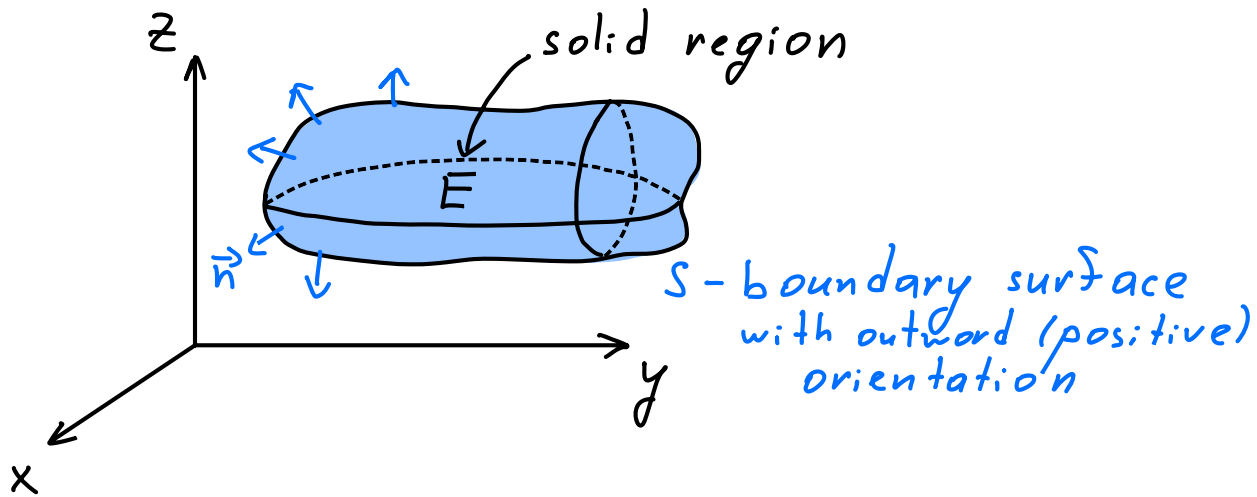
Stoke's THM:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S}$$

- line integral of (tangential component) of $\vec{F}$ along the boundary equals the flux of curl through the surface.

Notation: $C = \partial S$

Divergence theorem



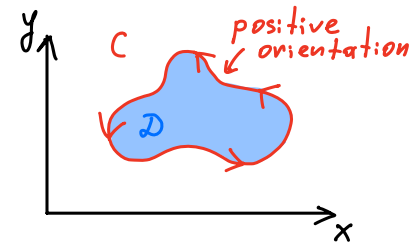
Divergence thm:

$$\underbrace{\iint_S \vec{F} \cdot d\vec{S}}_{\substack{\text{Flux of } \vec{F} \\ \text{through } S}} = \iiint_E \operatorname{div} \vec{F} dV$$

0. Green's thm:

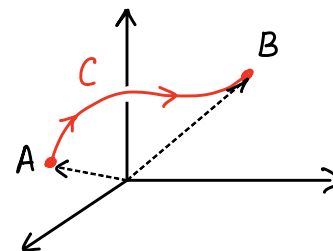
$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$\uparrow$ positively oriented



1. Fund. thm For line integrals:

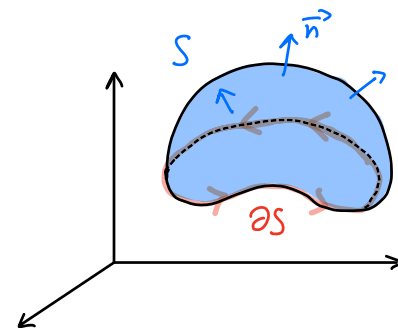
$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$



2. Stokes thm:

$$\iint_S \text{curl } \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

$\uparrow$ boundary curve



3. Divergence thm:

$$\iiint_E \text{div } \vec{F} dV = \iint_{\partial E} \vec{F} \cdot d\vec{S}$$

$\uparrow$ boundary surface

